**z-test for location**

A *hypothesis test* used to find out whether a *sample mean* $\bar{x}$ differs significantly (→ *significance*) from a known mean $\mu_0$. It uses the *standard normal distribution* to compute the improbability of the *observation* $\bar{x}$. When the sample size is small ($n < 30$), the *$t$-test for location* should be used instead. The *test statistic* ("$z$-score") used in the $z$-test is computed as follows:

$$z = \frac{\bar{x} - \mu_0}{SE} = \frac{\bar{x} - \mu_0}{\sqrt{s^2/n}}$$

where $s^2$ is the *sample variance*, $n$ is the sample size, and $SE = \sqrt{s^2/n}$ is the *standard error* of the sample.

The type of the $z$-test depends on its *null hypothesis*, $H_0$. When $H_0$ states $\bar{x} = \mu_0$, then the test is *two-tailed*, otherwise, when $H_0$ states either $\bar{x} \leq \mu_0$ or $\bar{x} \geq \mu_0$, it is one-tailed. The corresponding *critical regions* are shown in table ZTC.

| $H_0$ | Critical Region | Tail |
|---|---|---|
| $\bar{x} = \mu_0$ | $P(Z \leq -\lvert z\rvert) = \frac{\alpha}{2}$ and $P(Z \geq \lvert z\rvert) = \frac{\alpha}{2}$ | Both |
| $\bar{x} \leq \mu_0$ | $P(Z \geq z) = \alpha$ | Right |
| $\bar{x} \geq \mu_0$ | $P(Z \leq z) = \alpha$ | Left |

Figure **ZTC:** critical regions of the $z$-test; $\alpha$ is the desired level of significance

The $z$-score (in $\sigma$; see *standard deviation*) indicates the improbability of the observed mean given the null hypothesis. It can be converted to a *quantile* using the *cumulative distribution function* $F_Z$ of the standard normal distribution.

Example: A training program is supposed to boost the IQ of its participants. The average IQ is $\mu_0 = 100$, the mean IQ taken from a sample of 50 participants is $\bar{x} = 105$ with a variance of $s^2 = 450$, and the chosen *level of significance* is $\alpha = 0.05$. The test score is then

$$z = \frac{105 - 100}{\sqrt{450/50}} = \frac{5}{3} \approx 1.667$$

Because in this case the null hypothesis $H_0$ would be that the IQ has not increased (i.e. $\bar{x} \leq \mu_0$), the corresponding critical region would start at $0.95$. The quantile of $1.667$ is $F_Z(1.667) \approx 0.952$,

so the result is (barely) within the critical region.

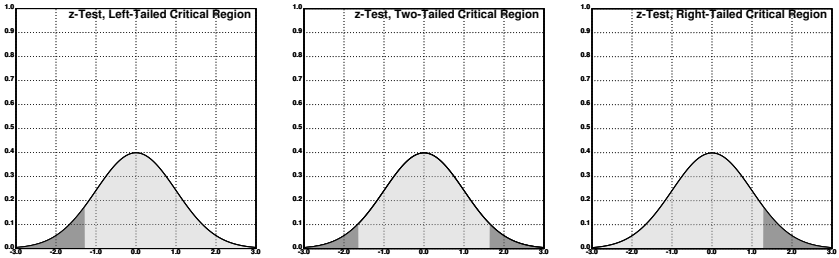


Figure **CRZ:** critical regions of the $z$-test Left: critical region in the left tail; middle: one critical region in each tail; right: critical region in the right tail